Shifted Legendre Polynomials For Solving Second Kind Fredholm Integral Equations

Shoukralla, E. S.
Department of Physics and EngineeringMathematics,
Faculty of Electronic Engineering,
Menoufia University
Menouf, Egypt
shoukralla@el-eng.menofia.edu.eg

Elgohary, H.
Department of Physics and EngineeringMathematics,
Faculty of Electronic Engineering,
Menoufia University
Menouf, Egypt
hany.elgouhari@el-eng.menofia.edu.eg

Morgan, M.
Department of Physics and Engineering Mathematics,
Faculty of Engineering and Technology,
German University, Egypt
Cairo, Egypt
Mariam.morgan@guc.edu.eg

***Abstract*—in this paper, present a computational method for solving Fredholm integral equations of the second kind. The method based on the application of the shifted Legendre polynomials in matrix forms. We create a technique for extracting the Legendre coefficients of each polynomial away so that each Legendre polynomial is rewritten in the form of its coefficient's matrix multiplied by the monomial basis function matrix. This technique significantly reduces the round off errors. By using this technique, the unknown and the data functions are expressed in two forms; each consists of three matrices. The kernel is approximated twice relevant to its two variables so that it is transformed into a form consists of a product of five matrices. By substituting by the approximate unknown function into the left and the right sides of the integral equation, we obtain an algebraic linear system of the equations without applying the collocation points. Moreover, we adapted the Gauss–Quadrature rule in an adjustment form and applied it for computing the resulted integrals. The convergence in the mean of the approximate solution and the kernel are proved. Additionally, the maximum norm error is studied, and it is found equal to zero. Numerical results are obtained for five examples to clarify the simplicity, efficiency, and reliability of the method. The obtained solutions are equal or rapidly converge to the exact solutions.**

Keywords—Fredholm integral equations; Legendre polynomials; Computational methods; approximate solution; error estimation.

I. Introduction

Integral equations arise in many scientific and engineering problems, for instance, image processing, inverse problems, bioengineering, electromagnetic, heat-conduction, radiation, astrophysics, potential and reactor theorems, quantum mechanics, diffraction problems, scattering in quantum mechanics [1-3].

Many initial and boundary value problems in engineering and physical science can be easily converted to Fredholm integral equations of specific kinds [4-7]. The reason for solving initial or boundary value problems through integral equation method is due to its ability to solve the problem easier, facilitate the proving of the uniqueness of the solution, empower to know more properties about the solution, and enable for the analytical treatments of the singularities of the solution of the original problems.

In [8-11], the authors provided methods for solving singular integral equations based on the application of orthogonal polynomials with different techniques. These methods can be applied for solving nonsingular integral equations. There were also some published articles [12-14] for solving Volterra integral equations of the second kind based on the barycentric Lagrange interpolation with various techniques, which are applicable for solving second kind Fredholm equations. But we try here to present a new method based on shifted Legendre polynomials in matrix form using some innovative ideas. There are valuable methods published for the numerical solutions of Fredholm integral equations of the second kind [15-24].

Abdullahet al., in [21] presented a method based on Toachard Polynomials (T-Ps) to solve linear Fredholm integral equation of the second kind. The authors in [22] presented an approach that is more efficient than the Nystrom method to find out a unique solution of the Fredholm integral equation of the second kind. Mohammad in [23] presented a method based on the use of B-spline quasi-affine tight framelet systems generated by the unitary and oblique extension principles. Maleknejad and Mahmoudi in [24] applied the hybrid Taylor polynomials and Block-pulse functions to estimate the solution of the linear Fredholm integral equation of the second kind.

The goal of this paper is to introduce a new approach for solving Fredholm integral eequations of the second kind. The proposed method based on the shifted Legendre polynomials in matrix form without applying the collocation method in order to reduce the solution to an algebraic linear system unlike most of the above-mentioned methods. We attempt to present a new simple, straightforward, and well-organized computational method that gives more accurate solutions regardless of the analyticity of the unknown functions and the kernels. Furthermore, we aspire to establish a method adequate for non-degenerate kernels, and singular equations. To reduce the round off errors and since integrating the monomial basis function is easier than integrating the Legendre polynomials, we extract the Legendre coefficients of each polynomial and put them in a separated matrix so that it can be expressed as a product of two matrices (the first matrix is known coefficients matrix and the second one is the monomial basis functions matrix).

First, we approximate the data function and the unknown function by using Legendre polynomials via monomial basis matrices of the same degree. The kernel is approximated twice in such a manner that the Legendre coefficients of the kernel which obtained from the first approximation are expanded once again about the second variable of the kernel by Legendre polynomial of the same degree. Thus, the Legendre coefficients of the kernel can be determined through double repeated integral formulas. Additionally, and to ensure the ease and accuracy of the obtained approximate repeated integrals, the Gauss-Legendre rule is adapted in an easy form and applied. Furthermore, and to dispense the application of collocation points to get an algebraic linear system equivalent to the required solution, we replacing the approximate unknown function twice into the considered integral equation. Thus, without applying collocation points and by utilizing some matrix operations, the required solution is transformed into the solution of an algebraic linear system.

Convergence in the mean of the unknown function, and the kernel are proved [25,26]. Besides some concepts and Euclidean norms of real analysis are applied to estimate the total approximation error [27,28]. The solutions of the illustrated four examples were exact and strongly converge to the exact solutions compared with the solutions obtained by the methods given in [21-24], which ensures the high accuracy and superiority of the presented numerical method.

II. DOUBLE APPROXIMATE KERNEL METHOD

Consider the linear Fredholm integral equation of the second kind

$$u(x) = f(x) + \int_a^b k(x,t)u(t)\,dt; \quad a \le x \le b \tag{1}$$

where $k(x,t)$ is the kernel, $f(x)$ is the data function, and $u(x)$ is the unknown function. We assume that $k(x,t)$ is not badly behaving function, and it is defined on the square $\{(x,t): a \le x, t \le b\}$ and $\int_a^b\int_a^b |k(x,t)|^2\,dxdt < M$; M is a real number. Moreover, we consider the two functions $u(x)$, $f(x)$ belong to $L_2(a,b)$ and $\max_{x\in[a,b]} f(x) = N$; N is a real number. The proposed computational method begins by redefining the infinite set of orthogonal Legendre polynomials $\{p_i(\xi)\}_{i=0}^{\infty}$ for $|\xi| \le 1$ to be applicable for approximating functions defined on $[a,b]$. In this context, the Legendre polynomials $\{p_i(x)\}_{i=0}^{\infty}$ for $x \in [a,b]$ can be defined by

$$p_0(x) = 1,$$
$$p_i(x) = \frac{1}{i!(a-b)^i}\frac{d^i}{dx^i}\left[(a-x)(x-b)\right]^i \;;\; i \ge 1\;;\; a \le x \le b \tag{2}$$

where the orthogonal property is defined by

$$\int_a^b p_i(x)\,p_j(x)\,dx = \frac{b-a}{2i+1}\delta_{ij};\qquad \delta_{ij} = \begin{cases} 1; & \text{if } i = j \\ 0; & \text{otherwise}\end{cases} \tag{3}$$

It is necessary here to mention that the existence and uniqueness theorem of Legendre polynomials confirm that if $f(x)$ is a piecewise continuous, and have a finite number of extrema, then the series $\sum_{n=0}^{\infty} u_i p_i(x)$; $x \in [a,b]$ converges to $f(x)$, where x is a continuous point and u_i are the Legendre coefficients. However, let $\tilde{u}_n(x)$ denotes the approximate unknown function, then we have

$$\tilde{u}_n(x) = \sum_{i=0}^{n} u_i p_i(x);\qquad u_i = \left(\frac{2i+1}{b-a}\right)\int_a^b u(x)\,p_i(x)\,dx \;;\; i = \overline{0,n} \tag{4}$$

In matrix form Eq. (4), takes the form

$$\tilde{u}(x) = \mathrm{UP}(x) \tag{5}$$

where $\mathrm{U} = [u_i]_{i=0}^{n}$ is the required unknown coefficients $1\times(n+1)$ row matrix and $\mathrm{P}(x) = [p_i(x)]_{i=0}^{n}$ is the column known matrix of order $(n+1)\times n$ whose entries $p_i(x)$ are defined by (2). Now, by expanding each Legendre polynomial $p_i(x); i = \overline{0,n}$ into Maclaurin polynomial of degree n we find that

$$\tilde{u}(x) = \mathrm{UAX}(x) \tag{6}$$

where

$$\mathrm{X}^T(x) = \left[x^i\right]_{i=0}^{n};\; \mathrm{A} = \left[a_{ij}\right]_{i,j}^{n}\;; a_{ij} = \frac{p_i^{(j)}(0)}{j!};\; i,j = \overline{0,n}; \tag{7}$$

Here $\mathrm{U} = [u_i]_{i=0}^{n}$ is the unknown required coefficients $1\times(n+1)$ row matrix, $\mathrm{X}^T(x)$ is monomial basis polynomial row matrix of order $1\times(n+1)$ and $\mathrm{A} = \left[a_{ij}\right]_{i,j}^{n}$ is a square known coefficients matrix of order $\left(n+1\right)\times\left(n+1\right)$ .By the same way, let $\tilde{f}_n(x)$ denotes the approximant function of the given data function $f\left(x\right)$, then we have

$$\tilde{f}\left(x\right)=\mathrm{FAX}\left(x\right) \tag{8}$$

Where $\mathrm{F}=\left[f_j\right]_{j=0}^{n}$ is the coefficient row matrix whose entries $f_i\left(x\right)$ can be computed by applying formula (4). Ultimately, sample $\tilde{k}_n(x,t)$ the approximate kernel obtained by approximating the original kernel $k(x,t)$ of Eq.(1) via Legendre polynomials with respect to the two variables x and t respectively. We begin by approximating the kernel with respect to x, and then with respect to t. Thus, we have

$$\tilde{k}_n(x,t)=\sum_{i=0}^{n}A_i(t)p_i(x);A_i(t)=\frac{2i+1}{b-a}\int_a^b k(x,t)p_i(x)\ dx \tag{9}$$

Once again, the Legendre coefficients $A_i(t)$, which are functions of the second variable t are approximated via Legendre polynomials of the same degrees to get

$$\begin{gathered}A_i(t)=\sum_{j=0}^{n}\alpha_{ij}p_j(t)\ \forall i=\overline{0,n};\\ \alpha_{ij}=\frac{2i+1}{b-a}\int_a^b A_i\left(t\right)p_j\left(t\right)dt;\ i,j=\overline{0,n}\end{gathered} \tag{10}$$

Substituting $A_i(t)$ that was given by Eq. (10) into Eq. (9), we define the $\left(n+1\right)\times\left(n+1\right)$ square matrix $\mathrm{C}^T=\left[c_{ij}\right]_{i,j=0}^{n}$ whose entries can be evaluated by

$$c_{ij}=\frac{(2i+1)(2j+1)}{(b-a)^2}\int_a^b\int_a^b k(x,t)p_i(x)p_j(t)dxdt;\ i,j=\overline{0,n} \tag{11}$$

Alternatively, and from the previous mathematical technique, the approximate kernel $\tilde{k}(x,t)$ can now be rewritten in the matrix form

$$\tilde{k}\left(x,t\right)=\mathrm{P}^T\left(t\right)\mathrm{CP}\left(x\right) \tag{12}$$

Now, expanding each Legendre polynomial $p_i\left(x\right),p_j\left(x\right);i=\overline{0,n}$ into Maclaurin polynomial of degree n we find that

$$\tilde{k}\left(x,t\right)=\mathrm{X}^T\left(t\right)\mathrm{A}^T\mathrm{CAX}\left(x\right) \tag{13}$$

Our strategically planning now may be summarized as follows: the two approximate functions $\tilde{u}_n\left(x\right),\tilde{k}_n(x,t)$ ofEqs. (6) and (13) are replaced with $u\left(x\right),k(x,t)$ of Eq. (1) to get $\tilde{u}_n\left(x\right)$ explicitly in the following form

$$\tilde{u}\left(x\right)=f\left(x\right)+\mathrm{UA\tilde{X}A}^T\mathrm{CAX}\left(x\right) \tag{14}$$

where the entries $\tilde{x}_{ij}$ of the $\left(n+1\right)\times\left(n+1\right)$ square matrix $\tilde{\mathrm{X}}=\left[\tilde{x}_{ij}\right]_{i,j=0}^{n}$ can be evaluated by

$$\tilde{x}_{ij}=\left[\frac{b^{i+j+1}-a^{i+j+1}}{i+j+1}\right]_{i,j=0}^{n};\tilde{\mathrm{X}}=\int_a^b\mathrm{X}\left(t\right)\mathrm{X}^T\left(t\right)dt \tag{15}$$

Accordingly,replace $\tilde{u}_n\left(x\right)$ again into both sides of Eq. (1) and considering Eq. (8), yields the followinglinear system of algebraic equations withoutapplying collocation points

$$\mathrm{UA\tilde{X}A}^T\mathrm{CAX}(x)-\ \mathrm{UA\tilde{X}A}^T\mathrm{CA\tilde{X}A}^T\mathrm{CAX}(x)=\mathrm{FA\tilde{X}A}^T\mathrm{CAX}(x) \tag{16}$$

Thus, we have

$$\mathrm{U}\left(\mathrm{I}-\mathrm{A\tilde{X}A}^T\mathrm{C}\right)=\mathrm{F} \tag{17}$$

Therefore, the solution of system (17) we get the unknown coefficients matrix U and by substituting into Eq. (6) we obtain the approximate unknown function $\tilde{u}_n(x)$. To improve the accuracy of the solution of the linear algebraic system (17), we adapted the Gauss-Legendre quadrature rule for computing andminimizing the round-off error of the repeated integrals c_{ij} that were given by Eq. (11). Let

$$\int_a^b f(x)dx=\sum_{s=1}^{m}\delta_s f\left(\omega_s\right) \tag{18}$$

Where

$$\begin{gathered}\delta_s=\frac{b-a}{\left(1-\alpha_s^2\right)\left(P'_m\left(\alpha_s\right)\right)^2};\omega_s=\rho_0\alpha_s+\rho_1;\rho_0=\frac{b-a}{2},\\ \rho_1=\frac{b+a}{2}\forall\ s=\overline{1,m}\ ;\sum_{s=1}^{m}\delta_s=2\end{gathered} \tag{19}$$

Here, the values $\left\{\alpha_s\right\}_{s=1}^{m}$ denote the zeros of Legendre polynomial $P_m\left(x\right)$ given by Eq. (2). Now, we attempt to make use of the adaptive m-nodes Gaussian quadrature rule of Eq. (18) to evaluate the integrals that weregiven by Eq. (11). In the context of this rule, the matrix C whose entries are the integrals $c_{ij};\ i,j=\overline{0,n}$ that were given by Eq. (11) is now transformed to the block matrix B of order $\left(n+1\right)\times\left(n+1\right)$, with blocks, b_{ij} of an order 1×1 such that

$$\mathrm{B}=\left[b_{ij}\right]_{i,j=0}^{n};b_{ij}=\mu_{ij}^2\mathrm{P}\left(\omega_s\right)\mathrm{Q}^T\tilde{\mathrm{P}}\left(\omega_s\right);\quad s=\overline{1,m} \tag{20}$$

where

$$\mu_{ij}^2 = \left(\frac{2i+1}{b-a}\right)\left(\frac{2j+1}{b-a}\right),$$

$\mathrm{P}(\omega_s) = [p_i(\omega_s)]_{s=1}^m \; \forall \; i = \overline{0,n}$ is a row matrix of order $1\times(n+1)$, $\tilde{\mathrm{P}}(\omega_s) = \left[p_j(\omega_s)\right]_{s=1}^m$ for each $j = \overline{0,n}$ is a column matrix of order $(n+1)\times 1$ and Q is the $(n+1)\times(n+1)$ matrix such that

$$\mathrm{Q} = \left[\delta_{sg} k_{sg}\right]_{s,g=1}^m, \text{where } \delta_{sg} = \delta_s \times \delta_g, \text{ and}$$

$$k_{sg} = k\left(\omega_s, \omega_g\right) \tag{21}$$

Similarly, the matrix F whose entries f_i can be computed by

$$f_i = \left(\frac{2i+1}{b-a}\right)\int_a^b f(x)\, p_i(x)\, dx \;;\; i = \overline{0,n} \tag{22}$$

is defined now by

$$\mathrm{F} = [f_i]_{i=0}^n \,;\; f_i = \sigma_i \sum_{s=1}^m \delta_s f(\omega_s)\, p_i(\omega_s) \;; \sigma_i = \mu_i \rho_0;\; i = \overline{0,n} \tag{23}$$

By solving system (17) we can find the unknown coefficients matrix U and thereby the approximate solution $\tilde{u}(x)$ from Eq. (6).

III. Convergence in the Mean and Error Estimation

Convergence in the mean of the approximate unknown function $\tilde{u}(x)$, the approximate data function $\tilde{f}(x)$, and the approximate kernel $\tilde{k}(x,t)$ are now proved. For convergence in the mean of $\tilde{u}(x)$ we have

$$\|u(x) - \tilde{u}(x)\|_2 = \int_a^b \left|u(x) - \tilde{u}(x)\right|^2 dx$$
$$= \int_a^b |u(x)|^2 dx - 2\int_a^b |u(x)\tilde{u}(x)| dx + \int_a^b |\tilde{u}(x)|^2 dx \tag{24}$$

From Eq. (5), and upon the application of the orthogonal property of Legendre polynomials given by Eq. (3), we find that

$$\left[|u(x)|^2\right] = \left[\sum_{i=0}^\infty a_i p_i(x) \sum_{j=0}^\infty a_j p_j(x)\right] = \mathrm{AP}(x) \tag{25}$$

where $\mathrm{A} = \left[a_i^2\right]_{i=0}^\infty$ an infinite row matrix is whose entries, a_i can be computed by Eq. (4) and $\mathrm{P}(x) = \left[p_i^2(x)\right]_{i=0}^\infty$ is an infinite column matrix of Legendre polynomials given by Eq. (2). Hence, we have

$$\left[\int_a^b |u(x)|^2 dx\right] = \mathrm{A}\tilde{\mathrm{P}}(x) \tag{26}$$

where $\tilde{\mathrm{P}}(x) = [\tilde{p}_i(x)]_{i=0}^\infty$ is an infinite column matrix whose entries $\tilde{p}_i(x)$ can be computed by the orthogonal property formula

$$\tilde{\mathrm{P}}_i(x) = \int_a^b p_i^2(x)\, dx = \frac{b-a}{2i+1};\, i \geq 0 \tag{27}$$

Hence, we have

$$\int_a^b |u(x)|^2 dx = (b-a)\sum_{i=0}^\infty \frac{a_i^2}{2i+1} \tag{28}$$

By the same way, we find that

$$\int_a^b |\tilde{u}(x)|^2 dx = (b-a)\sum_{i=0}^n \frac{u_i^2}{2i+1} \tag{29}$$

(22)

Consequently, by applying the Schwarz inequality, we have

$$\int_a^b |u(x)\tilde{u}(x)| dx \leq \left[\int_{-1}^1 |u(x)|^2 dx\right]^{\frac{1}{2}} \times \left[\int_{-1}^1 |\tilde{u}(x)|^2 dx\right]^{\frac{1}{2}} \leq \|u(x)\|_2 \cdot \|\tilde{u}(x)\|_2$$
$$\leq \left[(b-a)\sum_{i=0}^\infty \frac{a_i^2}{2i+1}\right]^{\frac{1}{2}} \times \left[(b-a)\sum_{i=0}^\infty \frac{u_i^2}{2i+1} t\right]^{\frac{1}{2}} \tag{30}$$

Furthermore, from Eqs. (28)- (30), we get

$$\lim_{n\to\infty} \|u(x) - \tilde{u}(x)\|_2 = 0 \tag{31}$$

For the convergence in the mean of $\tilde{k}(x,t)$, we find that

$$\int_a^b\int_a^b \left|\tilde{k}(x,t)\right|^2 dt dx = \left(\frac{b-a}{2i+1}\right)^2 \sum_{i=0}^n \sum_{j=0}^n c_{ij}^2 \,,$$
$$\int_a^b\int_a^b |k(x,t)|^2 dt dx = M^2 (b-a)^2 \tag{32}$$

and since

$$-2\int_a^b\int_a^b \left|k(x,t)\tilde{k}(x,t)\right| dt dx \leq -M^2(b-a)^2 - \left(\frac{b-a}{2i+1}\right)^2 \sum_{i=0}^n \sum_{j=0}^n c_{ij}^2 \tag{33}$$

(25)

Then

$$\lim_{n\to\infty} \left\|k(x,t) - \tilde{k}(x,t)\right\| = 0 \tag{34}$$

The goal now is devotedtoestimate the error of approximation. Sample the Legendre coefficient of the exact solution $u(x)$ by a_i, the approximate coefficient of the approximate solution $\tilde{u}(x)$ by $\tilde{u}_i$ and the total error of the approximation by $E_n(x)$. $E_n(x) = \|u_i - \tilde{u}_i\|_2$ where $\| \,.\, \|_2$ denotes the Euclidean norm in $\Re^2$. Returning to Eq. (1) andrewrite it in the form $u = Tu$, where operator T is defined by $Tu = f + Ku$. Similarly, the approximate integral equation associated to Eq. (2) with the approximate solution $\tilde{y}$ takes the form $T\tilde{u} = f + \tilde{K}\tilde{u}$, then we have

$$\|Tu - T\tilde{u}\|_2 = \|Ku - \tilde{K}\tilde{u}\|_2 = \left\| \int_a^b k(x,t)u(x)dx - \int_a^b \tilde{k}(x,t)\tilde{u}(x)dx \right\|_2 \tag{35}$$

Or

$$\|Ku - \tilde{K}\tilde{u}\| = \left[\int_a^b \left| \int_a^b k(x,t)u(x)dx - \int_a^b \tilde{k}(x,t)\tilde{u}(x)dx \right|^2 dt \right]^{\frac{1}{2}} \tag{36}$$

By Cauchy – Bunyakowski inequality, we have

$$\int_a^b \left| \int_a^b k(x,t)u(x)dx \right|^2 dt \le \int_a^b \|k(x,t)\|_2^2 \|u(x)\|_2^2 dt$$
$$= \|u(x)\|_2^2 \left[\int_a^b \left(\int_a^b |k(x,t)|^2 dx \right) dt \right] = M^2 (b-a)^2 \|u(x)\|_2^2 \tag{37}$$

and

$$\int_a^b \left| \int_a^b \tilde{k}(x,t)\tilde{u}(t)dx \right|^2 dt \le \int_a^b \|\tilde{k}(x,t)\|_2^2 \|\tilde{u}(x)\|_2^2 dt$$
$$= \|\tilde{u}(x)\|_2^2 \left[\int_a^b \left(\int_a^b |\tilde{k}(x,t)|^2 dx \right) dt \right] = \left(\frac{b-a}{2i+1} \right)^2 \sum_{i=0}^{n} \sum_{j=0}^{n} c_{ij}^2 \|\tilde{u}(x)\|_2^2 \tag{38}$$

And since,

$$\int_a^b \int_a^b \left| k(x,t)\tilde{k}(x,t) \right| dxdt$$
$$\le \frac{1}{2} \int_a^b \int_a^b |k(x,t)|^2 dxdt + \frac{1}{2} \int_a^b \int_a^b |\tilde{k}(x,t)|^2 dxdt \tag{39}$$

Then

$$-2\int_a^b \left| \int_a^b k(x,t)y(t)dt \times \int_a^b \tilde{k}(x,t)\tilde{y}(t)dt \right| dx$$
$$\le -M^2 (b-a)^2 \|u(x)\|_2^2 - \left(\frac{b-a}{2i+1} \right)^2 \sum_{i=0}^{n} \sum_{j=0}^{n} c_{ij}^2 \|\tilde{u}(x)\|_2^2 \tag{40}$$

Thus, we proved that

$$\lim_{n \to \infty} E_n(x) = \|Tu - T\tilde{u}\|_2 = 0 \tag{41}$$

IV. Computational Results of four illustrative examples

.we designed Matlab codes to solve Eq. (1) numerically using MATLAB 2014a. We solved the four similar examples, which are given in [21-24]. In all examples, we solved the system given by Eq. (17) to find the unknown matrix U and then substituting into Eq. (6) to obtain the approximate unknown function in an explicit form. The exact solution of example 1 is obtained for $n = 1$ and The CPU Time was 6.730626 seconds. The obtained solution is of course superior compared with the same example (no. 2) solved in [21]. For example, 2 the obtained numerical solutions for $n = 1$ is strongly converged to the exact solution with absolute errors $E(x) = 10^{-3}$ and the CPU Time was 3.083038 seconds. The solution gives better results than the solution obtained by the method described in [22]. The solution of example 3 is strongly converged to the exact solution for $n = 5$ with absolute error $10^{-13} \le E(x_i) \le 10^{-12}$ and the CPU Time equals to 13.220234 seconds, which is superior to the solution obtained by the method described in [24]. The solution of example 4 is strongly converged to the exact solution for (37) $n = 2$ with absolute errors $10^{-14} \le E(x_i) \le 10^{-9}$, which is superior to the solution obtained by the method described in [23]. The CPU Time was 3.219080 seconds. Tables 1, 2, and 3 show the comparison of the exact solutions $u(x_i)$ with the obtained numerical solutions $\tilde{u}(x_i)$ of examples 2, 3, and 4 respectively. In Figures 1, 2, and 3 the exact solutions are plotted against the numerical solutions for examples 2, 3, and 4 respectively.

Example 1.

$$u(x) = x + \int_{-1}^{1} (x^4 - t^4)u(t)dt$$

whose exact solution is given in [21] by $u(x) = x$. For $n = 1$, we get $U = [0 \quad 1]$ and $\tilde{u}(x) = x$.

Example 2. (39)

$$u(x) = 1 - e^{x-2} + e^{x-4} + \int_{-1}^{1} e^{x-t-3}u(t)dt$$

whose exact solution is given in [22] by $u(x) = 1$. Here, for $n = 1$, we have $U = [0.99950 \quad -0.00045]$ and $\tilde{u}(x) = 0.99950 - 0.00045x$.

Example 3.

$$u(x) = e^x + (1-e)x - 1 + \int_0^1 (x+t)u(t)dt$$

whose exact solution is given in [24] by $u(x) = e^x$. For $n = 5$ we have

$U = [\, 0.17423 \quad -2.01913 \quad -0.03356 \quad 1.35536 \quad -1.73926 \quad 1.34764 \,]$

and

$$\tilde{u}(x) = 1.88702x^5 + 0.15299x^4 - 0.65478x^3 - 0.42317x^2 + 0.75623x + 0.99999.$$

Example 4.

$$u(x) = 1 + \int_{-1}^{1} \left(xt + x^2t^2\right) u(t)\,dt$$

whose exact solution is given in [23] by $u(x) = 1 + \frac{10}{9}x^2$. Here, for $n = 2$, we have

$$U = \left[2.671700111 \quad -0.000000000 \quad -0.560589260\right]$$

and $\tilde{u}(x) = -0.8408835x^2 + 2.951994611$.

TABLE I.

x_i	Exact Solutions $u(x) = 1$	Numerical Solutions $\tilde{u}(x_i)$	Absolute Errors $E(x_i)$
-1	1	0.999951058172668	0.048941827332172 e-03
-0.89473	1	0.999903671546538	0.096328453461969 e-03
-0.78947	1	0.999856284920408	0.143715079591766 e-03
-0.68421	1	0.999808898294278	0.191101705721675 e-03
-0.57894	1	0.999761511668149	0.238488331851472 e-03
-0.47368	1	0.999714125042019	0.285874957981380 e-03
-0.36842	1	0.999666738415889	0.333261584111177 e-03
-0.26315	1	0.999619351789759	0.380648210240975 e-03
-0.15789	1	0.999571965163629	0.428034836370883 e-03
-0.05263	1	0.999524578537499	0.475421462500680 e-03
0.05263	1	0.999477191911369	0.522808088630589 e-03
0.15789	1	0.999429805285240	0.570194714760386 e-03
0.26315	1	0.999382418659110	0.617581340890294 e-03
0.36842	1	0.999335032032980	0.664967967020091 e-03
0.47368	1	0.999287645406850	0.712354593149889 e-03
0.57894	1	0.999240258780720	0.759741219279797 e-03
0.68421	1	0.999192872154590	0.807127845409594 e-03
0.78947	1	0.999145485528461	0.854514471539503 e-03
0.89473	1	0.999098098902331	0.901901097669300 e-03
1	1	0.999050712276201	0.949287723799097 e-03

Table1: A comparison of the exact solution $u(x) = 1$ of Example (2) with the obtained numerical solutions $\tilde{u}(x_i)$ of the presented method for $n = 1$ wit absolute errors $E(x_i)$.

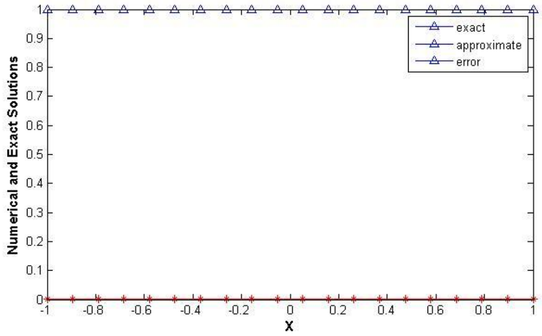

Fig. 1.The graph of the exact solution $u(x_i)$ versus the graphsof the numerical solutions $\tilde{u}(x_i)$ of Example (2) for $n = 1$ with the absolute errors $E(x_i)$.

TABLE II.

x_i	Exact Solutions $u(x_i)$	Numerical Solutions $\tilde{u}(x_i)$	Absolute Errors $E(x_i)$
0	1.000000000000000	0.999999999999521	0.4786171 e-12
0.05263	1.054041242591802	1.054041242591324	0.4787281 e-12
0.10526	1.111002941084471	1.111002941084567	0.9592326 e-13
0.15789	1.171042920543823	1.171042920543921	0.9792167 e-13
0.21052	1.234327535098344	1.234327535098866	0.5220268 e-13
0.26315	1.301032128860335	1.301032128860857	0.5220268 e-12
0.31578	1.371341521755806	1.371341521755327	0.4789502 e-12
0.36842	1.445450521609223	1.445450521608362	0.8610889 e-12
0.42105	1.523564463901954	1.523564463901333	0.6210587 e-12
0.47368	1.605899780699929	1.605899780699456	0.4729550e-12
0.52631	1.692684600326856	1.692684600326236	0.6201705 e-12
0.57894	1.784159379444528	1.784159379444127	0.4010125 e-12
0.63157	1.880577569291529	1.880577569291489	0.1426290 e-12
0.68421	1.982206317926315	1.982206317925836	0.4789502 e-12
0.73684	2.089327210420374	2.089327210419895	0.1615596 e-12
0.78947	2.202237049052355	2.202237049051877	0.4778399 e-12
0.84210	2.321248675664849	2.321248675664370	0.4791722 e-12
0.89473	2.446691838462353	2.446691838461874	0.4787281 e-12
0.94736	2.578914105652080	2.578914105651601	0.4791722 e-12
1	2.718281828459046	2.718281828458567	0.4791722 e-12

Table2: A comparison of the exact solution $u(x) = 1$ of Example (3) with the obtained numerical solutions $\tilde{u}(x_i)$ of the presented method for $n = 5$ wit absolute errors $E(x_i)$.

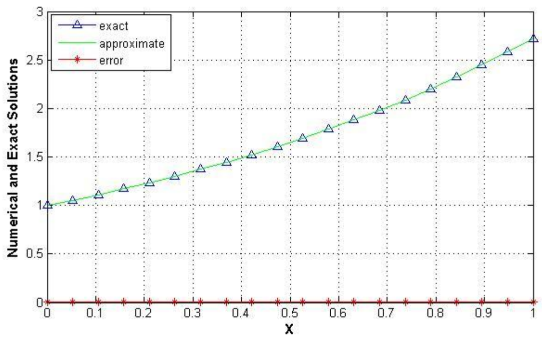

Fig. 2.The graph of the exact solution $u(x_i)$ versus the graphsof the numerical solutions $\tilde{u}(x_i)$ of Example (3) for $n = 5$ with the absolute errors $E(x_i)$.

TABLE III.

TABLE III: A COMPARISON OF THE EXACT SOLUTION $u(x)=1$ OF EXAMPLE (4) WITH THE OBTAINED NUMERICAL SOLUTIONS $\tilde{u}(x_i)$ OF THE PRESENTED METHOD FOR $n=2$ WIT ABSOLUTE ERRORS $E(x_i)$

Table III

x_i	Exact Solutions $u(x_i)$	Numerical Solutions $\tilde{u}(x_i)$	Absolute Errors $E(x_i)$
-1	2.111111111111111	2.111111111111966	8.5265128 E-14
-0.89473	1.889504462911665	1.889504462568584	3.4308112 E-10
-0.78947	1.692520775623269	1.692520775619571	3.6979309 E-12
-0.68421	1.520160049245922	1.520160049233527	1.2394974 E-11
-0.57894	1.372422283779625	1.372422283779318	3.0699887 e-12
-0.47368	1.249307479224377	1.249307479216351	8.0260243 e-12
-0.36842	1.150815635580179	1.150815635584825	4.6458393 e-12
-0.26315	1.076946752847030	1.076946752840474	6.5560890 e-12
-0.15789	1.027700831024931	1.027700831027609	2.6780800 e-12
-0.05263	1.003077870113881	1.003077870299889	2.88500912 e-9
0.05263	1.003077870113881	1.003077870117312	3.4310332 e-12
0.15789	1.027700831024931	1.027700831399879	2.97385916e-9
0.26315	1.076946752847030	1.076946752847591	5.6088467e-13
0.36842	1.150815635580179	1.150815635580446	3.5718983 e-11
0.47368	1.249307479224377	1.249307479224445	6.7945649 e-14
0.57894	1.372422283779625	1.372422283771589	8.0360163 e-12
0.68421	1.520160049245922	1.520160049249876	3.9539483 e-12
0.78947	1.692520775623269	1.692520775623308	3.9079850 e-14
0.89473	1.889504462911665	1.889504462911883	2.1782576 e-13
1	2.111111111111111	2.111111111111966	8.5265128 e-14

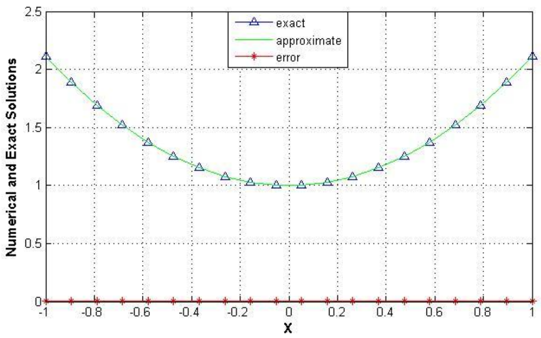

Fig. 3.The graph of the exact solution $u(x_i)$ versus thegraphsof the numerical solutions $\tilde{u}(x_i)$ of Example (2) for $n=2$ with the absolute errors $E(x_i)$

CONCLUSION

The Double approximate kernel method is presented for solving Fredholm Integral Equations of the Second kind. First, the unknown function, the given data function, and the kernel are approximated by using double - shifted Legendre polynomial of the same degree in matrix form. The extracting of the coefficients of Legendre polynomials and separated them from the monomial basis functions in matrix form significantly reduced the round off errors. The kernel is approximated twice with respect to the two arguments. The approximated unknown function is substituted twice into the integral equation, so that there was no need of collocation points, and the solution of the integral equation is transformed to the solution of an algebraic system of equations without applying the collocation points.. The convergence in the mean for both the unknown function and the kernel are proved. In order to remedy the complexity of the repeated integral formulas of Legendre coefficients an adjusted $m-$ nodes Gauss-Legendre quadrature rule was adapted though block matrices and applied for the evaluation of the Legendre coefficients integrals. The illustrated four examples demonstrate the authenticity and the efficiency of the presented method for lower degree polynomials regardless of the analyticity or smoothness of the unknown function and the kernel, thus ensures the Superiority of the presented method compared with other cited methods.

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