Numerical solution of singular Fredholm integral equations of the first kind using Newton interpolation

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Abstract

In this paper a computational technique is presented for the numerical solution of a certain potential-type singular Fredholm integral equation of the first kind with singular unknown density function, and a weakly singular logarithmic kernel. This equation is equivalent to the solution of the Dirichlet boundary value problem for Laplace equation for an open contour in the plane. The parameterization of the open contour facilitates the treatment of the density function's singularity in the neighborhood of the end-points of the contour, and the kernel's singularity. The unknown density function is replaced by a product of two functions; the first explicitly expresses the bad behavior of the density function, while the second is a regular unknown function, which will be interpolated using Newton interpolation in a matrix form. The singularity of the parameterized kernel is treated by expanding the two argument parametric functions into Taylor polynomial of the first degree about the singular parameter. Moreover, two asymptote formulas are used for the approximation of the kernel. In addition, an adaptive Gauss–Legendre formula, is applied for the computations of the obtained convergent integrals. Thus the required numerical solution is found to be equivalent to the solution of a system of algebraic equations. The numerical solution of the illustrated example is closer to the exact solution; which ensures the high accuracy of the presented computational technique.

1. Introduction

This paper is devoted to establish a computational technique for the numerical solution of a certain potential-type Fredholm integral equation of the first kind, whose unknown function (density) is singular in the neighborhood of the integration domain, and has a weakly singular kernel. In section 2, the reformulating of the Dirichlet problem for the Laplace equation of a mass distribution on an open contour in the plane is given. The equivalent potential-type Fredholm integral equation of the first kind is derived. In particular, Eq. (2) of section 2, arises in the fields of electron-optics, potential theory, and electromagnetism [1, 2] in the case of the replacement of a harmonic function by single-layer potentials. The finite energy condition, and the condition that the potential function tends to a constant at infinity are also imposed for the stability of the solution of the Dirichlet problem.

It should be noted that many methods [3-11] have been published for the solution of such equation, while the goal of the presented paper is to give an innovative approach for the numerical solution of this potential–type in the case of an open contour in the plane, in such a manner that the singularities of the density function disappear upon interpolating it via Newton Interpolation in matrix forms.

Actually, the approximation of the density function is based on replacing it by a product of two functions, the first is "badly behaved"; since it explicitly represents the singularity of the original singular density functions, while the second is a regular unknown function. The regular unknown function is approximated by Newton interpolating polynomial of degree n . Furthermore, the parameterization of the open contour facilitates the analytical treatment of the weak singularity of the kernel. The two argument parametric equations of the parameterized kernel are expanded into Taylor polynomial of the first degree about the singular parameter. Thus the singularities of the equation are entirely isolated. Consequently, the given data function is also expressed in Newton interpolant polynomial of the same degree n . In addition, an adaptive Gauss–Legendre formula is then applied for the computation of the obtained convergent integrals. Thus the required singular density function is found by solving a system of algebraic equations that minimizes the amount of calculations, and gives excellent results.

2. Formulation of the Dirichlet problem

Let $\Gamma \in \mathbb{R}^2$ be a simple, smooth, and open contour, that satisfies the Hölder condition $H^\alpha(\Gamma)$; $0 < \alpha \leq 1$, and Γ^+, Γ^- denote the 2-sides of Γ depending on the direction of the normal vector $\overrightarrow{n}$. Furthermore, let $\tilde{\Gamma} = \Gamma \cup \{a,b\}$, where $\{a,b\}$ are the end-points of Γ, and $u(x) \in \mathbb{R}^2 \left(\mathbb{R}^2 \setminus \tilde{\Gamma}\right)$ is the solution of the Dirichlet problem for the two Γ^+, Γ^- - dimensional Laplace equation $\Delta u(x) = 0$; $x \in \mathbb{R}^2 \setminus \tilde{\Gamma}$ under the Dirichlet condition $u^\pm(x) = u_o^\pm(x)$; $x \in \Gamma$, where $u_o^\pm(x) \in \mathbb{R}^{1,\alpha}(\Gamma)$, $0 < \alpha \leq 1$ is the given potential on both sides of Γ. In addition to the above mentioned conditions, it is provided that $\lim\limits_{|x| \to \infty} u(x) \to c$; c is a constant to ensure the stability of the solution, and the finite energy condition (edge condition) at the end-points of the open contour [12], where $C_i(\varepsilon)$; $i = 1,2$ denotes the two circles of radius $\varepsilon \Rightarrow 0$ with centers at the end-points a and b respectively. The method of boundary integral equations is applied so that the solution $u(x)$ of the Dirichlet problem for Laplace equation of a mass distribution on an open contour in the plane is derived by the expansion [1,12]

$$u(x) = \frac{1}{2\pi} \int_\Gamma \left(\tau(y) \ln \frac{1}{d(x,y)} - \frac{\partial}{\partial n_y} \ln \frac{1}{d(x,y)} \left(u_o^-(y) - u_o^+(y) \right) \right) ds_y \quad (1)$$

where $\tau(y)$ is the unknown density function of the mass distribution on Γ, that satisfies the integral formula $\int_\Gamma \tau(y) ds_y = 0$, and can be found as the solution of the equivalent Fredholm integral equation of the first kind

$$\frac{1}{2\pi} \int_\Gamma \tau(y) \ln \frac{1}{d(x,y)} ds_y = u_o(x) + \text{constant} ; x \in \Gamma \quad (2)$$

where

$$u_o(x) = \frac{1}{2}\left(u_o^-(y) - u_o^+(y)\right) - \frac{1}{2\pi}\int_{\Gamma}\frac{\partial}{\partial n_y}\ln\frac{1}{d(x,y)}\left(u_o^-(y) - u_o^+(y)\right)ds_y \quad (3)$$

3. Computational Technique

Consider the integral equation:

$$\int_{\Gamma}\phi(x)k(x,y)ds_x = f(y) \; ; y \in \Gamma \quad (4)$$

Here $\Gamma \in \Box^2$ is a simple open contour, that satisfies the Hölder condition $H^{\alpha}(\Gamma); \; 0 < \alpha \leq 1$, $k(x,y)$ is the kernel such that $k(x,y) = \ln\dfrac{1}{|d(x,y)|}$; where $d(x,y)$ denotes the distance between the two points on Γ, $\phi(x)$ is the undetermined singular density function which is defined on Γ, and $f(y) \in C(\Gamma)$ is the known given potential function. However, Γ is parameterized to get the 2-parametric equations that represents the point x on Γ; $x = x(\xi), y = y(\xi); \; a \leq \xi \leq b$;

$x'(\xi) \neq 0, y'(\xi) \neq 0; \; a < \xi < b$,

and the 2-parametric equations that represents the point y on Γ

$x = x(\tau), y = y(\tau); \; a \leq \tau \leq b$;

$x'(\tau) \neq 0, y'(\tau) \neq 0; \; a < \tau < b$,

Therefore, integral Eq. (1) is transformed to

$$\int_a^b \phi(\xi)J(\xi)K(\xi,\tau)d\xi = f(\tau) \; ; a \leq \tau \leq b \quad (5)$$

where ξ, τ are the parameters of the two points x, y on Γ, $J(\xi)$ is the Jacobian of the parameterized open contour Γ, such that

$J(\xi) = \sqrt{\left(x'(\xi)\right)^2 + \left(y'(\xi)\right)^2}$. Moreover, the function $\phi(\xi)$ is put as the product of two functions such as

$$\phi(\xi) = u(\xi) v(\xi) \tag{6}$$

where $u(\xi)$ is the regular unknown function, and $v(\xi) = \left(1 - \xi^2\right)^{\frac{-1}{2}}$ expresses the singular behavior of the unknown density function $\phi(\xi)$ near and at the end-points of the integration domain when $x \to \pm 1$. Interpolating $u(\xi)$ in Newton interpolant polynomial of degree n in a matrix form, we get

$$\left[u(\xi)\right] = \mathrm{U\,X} \tag{7}$$

where $\mathrm{U} = [u_i]_{i=0}^{n}$ is the row matrix of order $1 \times (n+1)$, whose entries u_i are the undetermined Newton's coefficients, and the matrix X is the column matrix of order $(n+1) \times 1$ such that

$$\mathrm{X}^T = \begin{bmatrix} 1 & (\xi - \xi_0) & (\xi - \xi_0)(\xi - \xi_1) & \cdots & (\xi - \xi_0)(\xi - \xi_1)\cdots(\xi - \xi_{n-1}) \end{bmatrix}$$

Now, by substituting Eq. (7), into Eq. (6) the unknown function becomes a regular function and takes the form

$$\left[\phi(\xi)\right] = \mathrm{U}\,\tilde{\mathrm{X}} \tag{8}$$

where $\tilde{\mathrm{X}}$ is the column matrix of order $(n+1) \times 1$

$$\tilde{\mathrm{X}} = \left[l_i(\xi)\right]_{i=0}^{n} \tag{9}$$

and

$$l_0\left(\xi\right) = \left(1\Big/\sqrt{1-\xi^2}\right) \; ; \; l_i\left(\xi\right) = \frac{1}{\sqrt{1-\xi^2}} \prod_{k=0}^{i-1} \left(\xi - \xi_k\right) \; \forall \; i \geq 1 \qquad (10)$$

Similarly, $f\left(\tau\right)$ is expanded in Newton interpolant polynomial of the same degree n in the matrix form

$$\left[f\left(\tau\right)\right] = F\Phi\left(\tau\right) \qquad (11)$$

Where

$$\Phi^T\left(\tau\right) =$$

$$\left[1 \quad \left(\tau - \tau_0\right) \quad \left(\tau - \tau_0\right)\left(\tau - \tau_1\right) \quad .... \quad \left(\tau - \tau_0\right)\left(\tau - \tau_1\right)..\left(\tau - \tau_{n-1}\right)\right]$$

and $F = \left[f_i\right]_{i=0}^{n}$ is the row matrix of order $1 \times \left(n+1\right)$, whose entries f_i can be calculated using the formula

$$f_i = \left(\Delta^i\left(f\left(\tau_0\right)\right)\Big/\left(i!\right)\left(h\right)^i\right); \; i = \overline{0,n} \; ; \; \left(\Delta^0\left(f\left(\tau_0\right)\right) = f\left(\tau_0\right)\right), \text{ where}$$

the step-size h can be evaluated by $h = \tau_{i+1} - \tau_i ; \; i = \overline{0,n-1}$.

Moreover, the logarithmic singular kernel $k\left(x,y\right) = ln\dfrac{1}{\left|d\left(x,y\right)\right|}$, where $d\left(x,y\right)$ is the distance from the parametric point $x\left(x\left(\xi\right),y\left(\xi\right)\right)$ to the parametric point $y\left(x\left(\tau\right),y\left(\tau\right)\right)$, can be rewritten in the form

$$k\left(\xi,\tau\right) = ln\frac{1}{\left|\sqrt{\left(x\left(\tau\right) - x\left(\xi\right)\right)^2 + \left(y\left(\tau\right) - y\left(\xi\right)\right)^2}\right|} \; ; \; a \leq \xi \leq b \qquad (12)$$

From Eq. (12) it turns out that the singularities of $k\left(x,y\right)$ occurs when $\xi \to \tau$, and to remedy this difficulty and treat this singular behavior, both $x\left(\xi\right), y\left(\xi\right)$ are approximated using Taylor polynomial of the first

degree about the singular parameter $\xi = \tau$. Thus, the following approximations are obtained

$$x(\xi) = x(\tau) + (\xi - \tau)x'(\tau) \; ; \; y(\xi) = y(\tau) + (\xi - \tau)y'(\tau) \tag{13}$$

By substituting Eq. (13) into Eq. (12), we obtain

$$k(\xi,\tau) = ln\frac{1}{R|\xi - \tau|} \; ; \; a \le \xi \le b, \; R = \sqrt{(x'(\tau))^2 + (y'(\tau))^2} \tag{14}$$

Finally, substituting Eqs. (8), (11), and (14) into Eq. (2), gives

$$U H(\tau) = F\Phi(\tau) \tag{15}$$

Here, $H(\tau) = \left[h_i(\tau)\right]_{i=0}^{n}$ is the column matrix of order $(n+1) \times 1$, where $h_i(\tau)$ can be evaluated by the integral formula

$$h_i(\tau_i) = \int_a^b l_i(\xi)ln\frac{1}{R|\xi - \tau_i|}d\xi;$$

$$R = \sqrt{(x'(\tau_i))^2 + (y'(\tau_i))^2} \; ; a \le \tau_i \le b, i = \overline{0,n} \tag{16}$$

Now, if the matrix equation (15) is satisfied at the n-collocation points $\{\tau_i\}_{i=0}^{n} \; ; \; a \le \tau_i \le b$, the following algebraic linear system is obtained

$$U H(\tau_i) = F\Phi(\tau_i) \quad \forall \; i = \overline{0,n} \tag{17}$$

Now, the apdative m – nodes Gauss-Legendre formula

$$\int_a^b f(x)dx = \sum_{s=1}^{m}\delta_s f(\omega_s) \tag{18}$$

where the weights δ_s, and the nodes ω_s are given by

$$\delta_s = \frac{b-a}{\left(1 - \alpha_s^2\right)\left(P'_m(\alpha_s)\right)^2} \quad , \tag{19}$$

$$\omega_s = \frac{b-a}{2}\alpha_s + \frac{b+a}{2} \quad \forall \ s = \overline{1,m} \ ; \ \sum_{s=1}^{m} \delta_s = 2$$

and α_s are the roots of Legendre polynomial of degree m ; that is defined on $[-1, 1]$, is applied for the computing of the integrals $h_i(\tau_i)$ since they became proper due to the analytical treatment based on the nature of Newton interpolating polynomial. Thus, the numerical solution of Eq. (4) is now equivalent to the solution of the linear system of algebraic equations given by Eq. (17). The unknown matrix U can be found, and thereby, the unknown density function $\phi(\xi)$ can be obtained by substituting into Eqs. (6) and (7).

4. Computational Results

To verify the presented technique, the numerical solution of a certain potential-type Fredholm integral equation of the first kind with a singular density (unknown function), and a singular logarithmic kernel (Eq. (20)) is obtained. However, the solution is found to be closer to the exact one. Tables of the obtained numerical solutions and the graphs are illustrated with the absolute error estimation to show the high accuracy of the presented method. Consider

$$\int_{-1}^{1} f(x)\ln\frac{1}{|x-y|}dx = 1 \ ; \ -1 \le y \le 1 \tag{20}$$

whose exact solution [5] is given by

$$f(x) = \frac{1}{(\pi \ln(2))}\left(1 - x^2\right)^{\frac{-1}{2}} \ ; \ -1 < x < 1 \tag{21}$$

From table 1, it can be observed that the obtained solutions by the presented technique are closer to the exact one. In table 2, the absolute error estimation shows the efficiency of the proposed method. The number of collocation points are only two points ($n = 2$), while the number of

Gauss–Legendre's nodes are $m = 8, 10,$ and 12. Here, $E_i(x_i)$ denotes the exact solution.

Table 1: Numerical solution using Newton interpolation.

i	x_i	$E_i(x_i)$	$m = 8$	$m = 10$	$m = 12$
0	-0.9000	1.0535	1.2016	1.2720	1.1793
1	-0.8000	0.7654	0.8434	0.8917	0.8337
2	-0.7000	0.6430	0.6864	0.7248	0.6837
3	-0.6000	0.5740	0.5954	0.6277	0.5975
4	-0.5000	0.5303	0.5362	0.5644	0.5422
5	-0.4000	0.5011	0.4958	0.5210	0.5050
6	-0.3000	0.4814	0.4679	0.4908	0.4801
7	-0.2000	0.4687	0.4493	0.4704	0.4642
8	-0.1000	0.4615	0.4383	0.4579	0.4558
9	0.0000	0.4592	0.4339	0.4524	0.4540
10	0.1000	0.4615	0.4358	0.4535	0.4585
11	0.2000	0.4687	0.4442	0.4614	0.4697
12	0.3000	0.4814	0.4601	0.4769	0.4885
13	0.4000	0.5011	0.4849	0.5017	0.5167
14	0.5000	0.5303	0.5219	0.5390	0.5576
15	0.6000	0.5740	0.5767	0.5947	0.6175
16	0.7000	0.6430	0.6620	0.6816	0.7098
17	0.8000	0.7654	0.8102	0.8329	0.8693
18	0.9000	1.0535	1.1502	1.1811	1.2343

Table 2: Absolute error estimation, where $E_1, E_2,$ and E_3 are the absolute errors for $m = 8, 10,$ and 12 respectively.

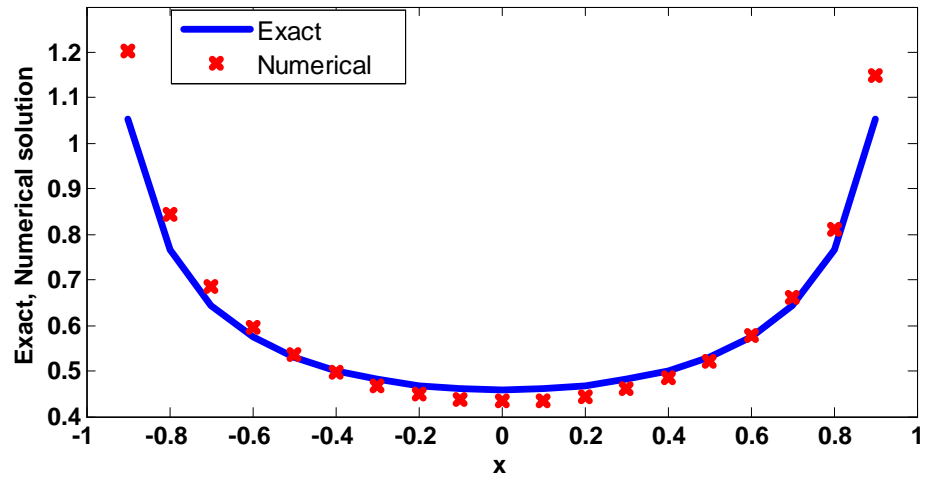| i	x_i	E_1	E_2	E_3
0	0.0000	0.0253	0.0068	0.0052
1	0.1000	0.0258	0.0080	0.0030
2	0.2000	0.0244	0.0073	0.0010
3	0.3000	0.0213	0.0045	0.0071
4	0.4000	0.0161	0.0007	0.0156
5	0.5000	0.0084	0.0087	0.0273
6	0.6000	0.0027	0.0207	0.0435
7	0.7000	0.0190	0.0385	0.0667
8	0.8000	0.0448	0.0676	0.1039
9	0.9000	0.0967	0.1275	0.1808

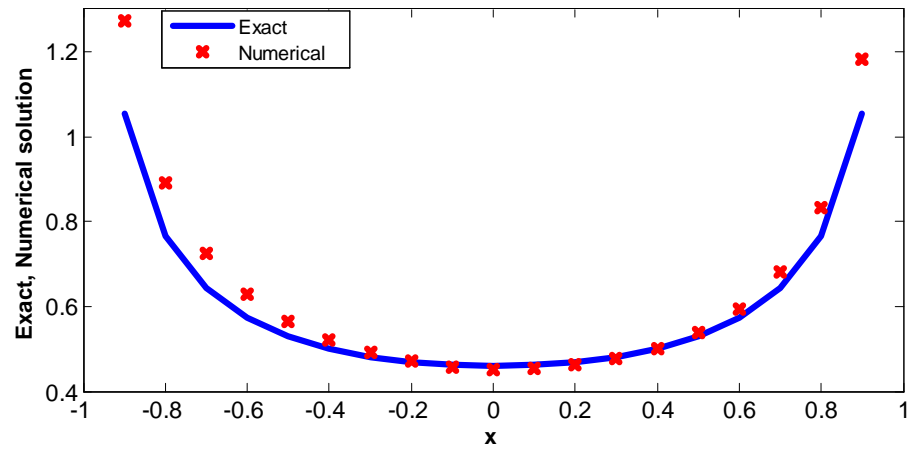Fig. 1: The presented technique for $m = 8$.
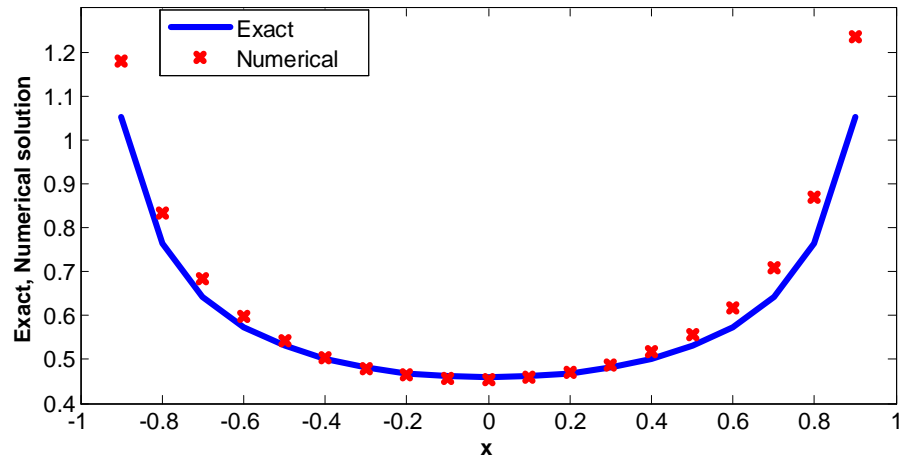

Fig. 2: The presented technique for $m = 10$.

Fig. 3: The presented technique for $m = 12$.

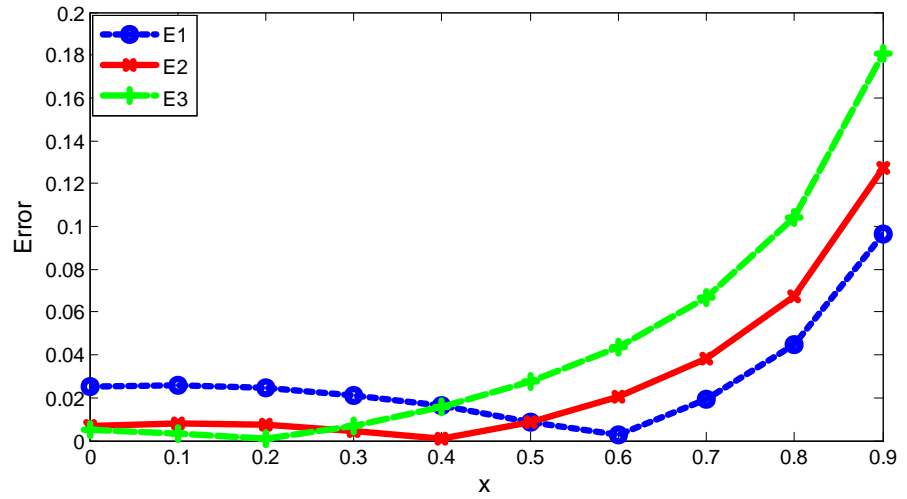Fig. 4: The absolute errors E_1, E_2, and E_3 for $m = 8,\ 10,\ \text{and}\ 12$ respectively.

5. Conclusion

A computational technique has been investigated for the numerical solution of a certain potential-type Fredholm integral equation of the first kind with a singular unknown (density) function at the end-points of the integration domain, and has a weakly singular logarithmic kernel. The presented technique is based on Newton interpolation in a matrix form, and the analytical treatment of the singularities without changing of the variables. The singularity of the unknown function was treated by considering it as a product of two functions. The first was given in a closed form that expresses its singular behavior, while the second, which is a regular function, was interpolated using Newton interpolation in a matrix form. The singularity of the kernel was treated analytically by expanding the two argument parametric functions of the parameterized kernel via Taylor polynomial of the first degree about the singular parameter. Furthermore, an adaptive Gauss–Legendre formula was applied, and matrix algebra was utilized, in such a manner that the unknown density function was found by solving a linear system of algebraic equations. The given numerical example has demonstrated the high accuracy of the presented technique.

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الملخص باللغة العربية

يقدم هذا البحث أسلوب جديد لحل معادلة فريدهولم التكاملية من النوع الأول والشاذة في نواتها وأيضاً في الدالة المجهولة بالقرب من حدود التكامل. حيث تم تقريب الدالة المجهولة فقط وذلك باستخدام كثيرات حدود نيوتن، مع المعالجة التحليلية لشذوذ النواة، أما شذوذ الدالة المجهولة فقد أمكن معالجته عن طريق إعتبار الدالة المجهولة هي حاصل ضرب دالتين، الأولى تحتوي علي السلوك الشاذ، أما الثانية فهي الداله المنتظمة التي تم تقريبها باستخدام كثيرات حدود نيوتن. وبتطبيق صيغة جاوس ليجندر لحساب التكاملات يتحول النظام الي مجموعة من المعادلات الجبرية حيث يَسهُل حَلُّها للحصول علي الدالة المجهولة. وقد أظهرت النتائج التي حصلنا عليها قوة الطريقة المُقتَرَحَة، حيث تقاربت حلول الطريقة المقترحة مع الحلول المضبوطة.